MOTION OF A BODY WITH A CAVITY FILLED WITH A VISCOUS FLUID AT LARGE REYNOLDS NUMBERS

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PMM Vol. 30, No. 3, 1966, pp. 476-494

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(Received October 22, 1965)

The problem of motion of a rigid body with a cavity filled with a viscous fluid has been the subject of a number of papers (for example [1 to 5]). An analysis of small oscillations of a viscous fluid at large Reynolds numbers, based on the idea of the boundary value, is given in [3]. The same method is used in [4], where small oscillations of a pendulum with an axially symmetric cavity filled with a viscous fluid at large Reynolds numbers were considered.

In this paper we consider the motion of a rigid body with a cavity of an arbitrary form completely filled with a viscous fluid. Two assumptions are made: (1)-that the amplitude of motion is small, so that the Navier-Stokes equations can be linearised, and (2)-that the Reynolds number is large. The linearised Navier-Stokes equations are solved here, similarly to [3 and 4], by the method of the boundary value. The viscosity of the fluid in the cavity introduces additional terms into the equations of motion of the rigid body. It is shown that the dependence of these terms on the form of the cavity is expressed by a symmetric tensor, similar to the mass tensor, and representing dissipated energy. Components of this tensor are expressed by the Joukowski potentials only [1], i.e. by the solution of the problem of motion of a perfect fluid in a cavity of a given form.

The computation of the motion of a rigid body with its cavity completely filled with a viscous fluid, with the assumptions stated above, requires therefore: (1) the determination of the Joukowski potentials for a given cavity (these are already known for a number of cavity forms); (2) the determination, by means of integration of the Joukowski potentials, of the associated moments of inertia and of the components of the tensor representing dissipated energy; and (3) the solution of equations of motion of the rigid body containing the additional terms. Problems (1) and (2) can be solved in advance for a large class of cavity forms. The process of solving this problem is, therefore, only a little more complicated than in the case of a cavity filled with a perfect fluid. [1]. We should note that similar results were arrived at in [5] by an entirely different method.

General equations of motion of a fluid filled body are derived for stated assumptions.

Certain specific forms of cavity are analysed. The analysis relates to small oscillations of a body having its cavity filled with a viscous fluid.

1. Analysis of the Navier-Stokes equations. The motion of a rigid body with a simply connected cavity D, completely filled with a viscous incompressible fluid of density ρ_0, is considered (fig. 1).

The equations of motion of the fluid have the form

$$\frac{d\mathbf{u}_a}{dt} = -\frac{\nabla p}{\rho_0} + \nu\Delta\mathbf{u}_a - \dot{\nabla} U, \qquad \operatorname{div}\mathbf{u}_a = 0 \tag{1.1}$$

Here, t denotes time, $\mathbf{u}_a$ is the absolute velocity of fluid particles, p is the pressure, ν is the kinematic viscosity, while U ($\mathbf{r}$, t) is the assumed potential of mass forces. The radius vector $\mathbf{r}$ shall always have its origin at point O, arbitrarily chosen, but rigidly connected with the body. The absolute radius vector of the pole O is denoted by $\mathbf{R}_0$.

Let us introduce the velocity $\mathbf{u} = \mathbf{u}_a - d\mathbf{R}_0 / dt$ relative to pole O, and re-write Equations (1.1) as follows

$$\mathbf{u}_t + (\mathbf{u}\nabla)\,\mathbf{u} = -\nabla q + \nu\Delta\mathbf{u}, \qquad \operatorname{div}\mathbf{u} = 0, \qquad q = \frac{p}{\rho_0} + U + \mathbf{r}\cdot\frac{d^2\mathbf{R}_0}{dt^2} \tag{1.2}$$

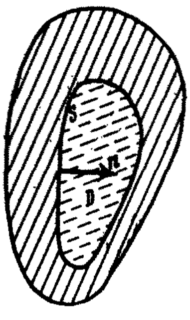

FIG. 1

Here index t denotes a partial derivative with respect to time, and q is a new function which is to be determined. The boundary and initial conditions for the system of equations (1.2) are given by

$$\mathbf{u} = \boldsymbol{\omega} \times \mathbf{r} \quad \text{on } S \tag{1.3}$$

$$\mathbf{u} = \mathbf{u}_0(\mathbf{r}) \quad \text{when } t = t_0 \tag{1.4}$$

Here, $\boldsymbol{\omega}(t)$ is the angular velocity of the rigid body, and S is the boundary of cavity D. The initial velocity $\mathbf{u}_0(\mathbf{r})$ is subject to conditions

$$\operatorname{div}\mathbf{u}_0 = 0 \quad \text{in } D, \quad \mathbf{u}_0 = \omega(t_0) \times \mathbf{r} \quad \text{on } S \tag{1.5}$$

Let L be a characteristic dimension of the cavity, and T a characteristic unit of time, for example, the oscillation period of the body. The Reynolds number is assumed to be large

$$R = L^2\nu^{-1}T^{-1} \gg 1 \tag{1.6}$$

We further assume that $|\mathbf{u}_0| \sim \omega L$. For the purpose of linearising equations (1.2), we shall assume that everywhere $|(\mathbf{u}\nabla)\,\mathbf{u}| \ll |\mathbf{u}_t|$.

As outside the boundary layer the order of magnitude is $|\mathbf{u}| \sim \omega L$, and $\nabla \sim L^{-1}$, it follows that $|\mathbf{u}\nabla| \sim \omega$. Within the boundary layer various components of vectors $\mathbf{u}$ and ∇ are of different orders of magnitude (see below), but here also $|\mathbf{u}\nabla| \sim \omega$.

Therefore, the condition of linearisation has the form

$$\omega T \ll 1 \tag{1.7}$$

The dimensionless parameter ωT is of the order of the angular amplitude of oscillation of the body. We assume in the following that conditions of (1.6) and (1.7) are fulfilled.

Condition (1.7) permits the linearisation of Equations (1.2)

$$\mathbf{u}_t = -\nabla q + \nu\Delta \mathbf{u}, \qquad \operatorname{div} \mathbf{u} = 0 \tag{1.8}$$

For convenience of notation we shall select units of length and time, such that $L \sim 1$ and $T \sim 1$. Then the condition (1.6) means that $\nu \ll 1$. We shall use the boundary value method [6] for solving the boundary problem (1.8), (1.3) and (1.4) for small values of parameter ν in higher derivatives. We assume

$$\begin{aligned} \mathbf{u} &= \mathbf{v} + \mathbf{w}, \qquad \mathbf{v} = \mathbf{v}^\circ + \nu^{1/2}\mathbf{v}^1 + \nu\mathbf{v}^2 + \ldots \\ q &= g + h, \qquad g = g^\circ + \nu^{1/2} g^1 + \nu g^2 + \ldots \end{aligned} \tag{1.9}$$

Upper indices denote the order of approximation. Here $\mathbf{w}$ and h are functions typical of boundary value problems [6], which rapidly tend to zero with the increasing distance from the cavity walls.

We select functions $\mathbf{v}^0$ and g^0 so, as to satisfy the equations of motion of a perfect fluid, the condition of absence of flow through the wall, and the initial conditions (1.4). The boundary problem for $\mathbf{v}^0$ and g^0 is expressed by

$$\begin{aligned} \mathbf{v}_t^\circ &= -\nabla g^\circ, \qquad \operatorname{div} \mathbf{v}^\circ = 0 \text{ in } D \\ \mathbf{v}^\circ\mathbf{n} &= (\boldsymbol{\omega} \times \mathbf{r}) \cdot \mathbf{n} \quad \text{on } S, \qquad \mathbf{v}^\circ = \mathbf{u}_0 \quad \text{when } t = t_0 \end{aligned} \tag{1.10}$$

Here $\mathbf{n}$ is the unit vector of the inward normal to S. In order to solve problem (1.10) we shall, first of all, determine vector $\mathbf{a}$ ($\mathbf{r}$) in D from

$$\operatorname{curl} \mathbf{a} = \operatorname{curl} \mathbf{u}_0, \qquad \operatorname{div} \mathbf{a} = 0 \quad \text{in } D, \ \mathbf{an} = 0 \text{ on } S \tag{1.11}$$

It is known [7], that the conditions (1.11) uniquely determine vector $\mathbf{a}$. It follows from (1.10) and (1.11) that

$$[\operatorname{curl}(\mathbf{v}^\circ - \mathbf{a})]_t = 0, \qquad \operatorname{curl}(\mathbf{v}^\circ - \mathbf{a}) = 0 \text{ when } t = t_0$$

Consequently, within the space D and for $t \geqslant t_0$, the vector $\mathbf{v}^0 - \mathbf{a}$ is a potential one. We can therefore write $\mathbf{v}^\circ = \mathbf{a} + \nabla \varphi^\circ$, and arrive, through (1.10) and (1.11), at the Neumann's problem for the function φ° $(\mathbf{r}, t)$

$$\Delta\varphi^\circ = 0 \quad \text{in } D, \qquad \partial\varphi^\circ / \partial n = (\boldsymbol{\omega} \times \mathbf{r}) \cdot \mathbf{n} \text{ on } S \tag{1.12}$$

Conditions (1.12) determine the harmonic function φ° with an approximation to an arbitrary function of time. From the first equation of (1.10) we have $g^\circ = -\varphi_t^\circ + C^\circ(t)$, where $C^0(t)$ is an arbitrary function. It is clear that, with account being taken of (1.11) and (1.12), functions $\mathbf{v}^\circ = \mathbf{a} + \nabla\varphi^\circ$ and g^0 satisfy the equations and boundary conditions (1.10). In order to check the initial condition of (1.10), it will be necessary to prove that vector

$$\mathbf{a}_1(\mathbf{r}) = \mathbf{v}^\circ(\mathbf{r}, t_0) - \mathbf{u}_0(\mathbf{r}) = \mathbf{a}(\mathbf{r}) + \nabla\varphi^\circ(\mathbf{r}, t_0) - \mathbf{u}_0(\mathbf{r})$$

equals zero within space D. By virtue of (1.11), (1.12) and (1.5) we have

$$\operatorname{curl} \mathbf{a}_1 = 0, \qquad \operatorname{div} \mathbf{a}_1 = 0 \quad \text{in } D, \qquad \mathbf{a}_1\mathbf{n} = 0 \quad \text{on } S$$

Hence, $\mathbf{a}_1 \equiv 0$ in D.

Thus, the determination of $\mathbf{v}^\circ$ requires the solution of the problems (1.11) and (1.12). We note that for the most important case of motion starting from rest, we have $\mathbf{u}_0 = 0$ and $\mathbf{a} = 0$. The solution of problem (1.12) can be expressed by the Joukowski potentials [1 and 2]

$$\varphi^\circ = \omega_1\Phi_1 + \omega_2\Phi_2 + \omega_3\Phi_3 \qquad (1.13)$$

Here, ω_i $(i = 1, 2, 3)$ are the projections of the vector $\boldsymbol{\omega}$ on the axes of an arbitrary system of coordinates $Ox_1x_2x_3$, rigidly attached to the body. Functions Φ_i satisfy boundary conditions

$$\Delta\Phi_i = 0 \text{ in } D, \quad \partial\Phi_i/\partial n = (\mathbf{r} \times \mathbf{n})\cdot\mathbf{e}_i \text{ on } S \; (i = 1,2,3) \qquad (1.14)$$

where $\mathbf{e}_i$ is the unit vector on the axis Ox_i.

Functions $\mathbf{v}^1$ and g^1 will be made to conform to the equations and to satisfy the following initial conditions.

$$\mathbf{v}_t^1 = -\nabla g^1, \; \operatorname{div}\mathbf{v}^1 = 0 \text{ in } D, \qquad \mathbf{v}^1 = 0 \text{ when } t = t_0$$

It follows from this that curl $\mathbf{v}^1 \equiv 0$, in D, for $t \geqslant t_0$, and we can, therefore, assume that $\mathbf{v}^1 = \nabla\varphi^1$. We then have for the functions φ^1, and g^1

$$\triangle\varphi^1 = 0, \qquad g^1 = -\varphi_t^1 + C^1(t) \text{ in } D$$

Here, $C^1(t)$ is an arbitrary function. We note that functions

$$\mathbf{v} = \mathbf{v}^\circ + \nu^{1/2}\mathbf{v}^1, \qquad g = g^\circ + \nu^{1/2}g^1$$

satisfy the equation of motion (1.8) in D, with an error of the order of γ, and satisfy the equation of continuity and the initial condition (1.4), exactly.

To satisfy the boundary adhesion condition (1.3) we assume, in conformity with (1.9), that

$$\mathbf{u} = \mathbf{v}^\circ + \nu^{1/2}\mathbf{v}^1 + \mathbf{w} = \mathbf{a} + \nabla\varphi^\circ + \nu^{1/2}\nabla\varphi^1 + \mathbf{w}, \; g = g^\circ + \nu^{1/2}g^1 + h \qquad (1.15)$$

Terms of the order of ν and higher appearing in the expansion (1.9), were neglected. Taking into account the equations and initial conditions which functions $\mathbf{v}^\circ$, $\mathbf{v}^1$, g°, g^1, and g^1 fulfill, and substituting (1.15) into (1.8) and (1.4), we derive the equations and the initial condition for functions $\mathbf{w}$, h.

$$\mathbf{w}_t = -\nabla h + \nu\triangle\mathbf{w}, \qquad \operatorname{div}\mathbf{w} = 0 \quad \text{в } D, \; \mathbf{w} = 0 \text{ when } t = t_0 \qquad (1.16)$$

We see from (1.3) and (1.15), that the following boundary condition must also be fulfilled

$$\mathbf{w} + \nu^{1/2}\nabla\varphi^1 = \boldsymbol{\omega} \times \mathbf{r} - \mathbf{v}^\circ \quad \text{on } S \qquad (1.17)$$

We introduce a system of curvilinear orthogonal coordinates ξ, η and ζ such that the boundary surface S coincides with the surface $\zeta = 0$, and that any point within S has the value $\zeta > 0$. We denote in this system the corresponding Lamé coefficients by H_ξ, H_η, H_ζ

(without affecting the generality, we assume that $H_\zeta = 1$ for $\zeta = 0$), and by w_ξ, w_η, w_ζ the components of the vector **w** in the above system.

We write down the equations of motion and of continuity (1.16) in the system of coordinates ξ, η, ζ. We then substitute

$$\zeta = \alpha \nu^{1/2}, \qquad w_\zeta = \nu^{1/2} w_\alpha$$

and seek expressions for w_ξ, w_η, and w_α in terms of ξ, η, α and t in the region D_1 of the boundary layer, adjacent to surface S. In the region D_1 we have $\zeta \sim \nu^{1/2}$, and $\alpha \sim 1$, and hence

$$H_\xi = H_\xi^\circ + O(\nu^{1/2}), \qquad H_\eta = H_\eta^\circ + O(\nu^{1/2}), \qquad H_\zeta = 1 + O(\nu^{1/2})$$

where H_ξ°, and H_η° are the values of Lamé coefficients on the surface S, i.e. where $\zeta = \alpha = 0$. We shall simplify the equations of motion and continuity in the region D_1 by taking into account these assumptions, and omitting in these equations terms of the order of $\nu^{1/2}$

$$\frac{\partial h}{\partial \alpha} = 0, \qquad \frac{\partial w_\xi}{\partial t} = -\frac{\partial h}{\partial \xi} + \frac{\partial^2 w_\xi}{\partial \alpha^2}, \qquad \frac{\partial w_\eta}{\partial t} = -\frac{\partial h}{\partial \eta} + \frac{\partial^2 w_\eta}{\partial \alpha^2}$$
$$\frac{\partial (H_\eta^\circ w_\xi)}{\partial \xi} + \frac{\partial (H_\xi^\circ w_\eta)}{\partial \eta} + H_\xi^\circ H_\eta^\circ \frac{\partial w_\alpha}{\partial \alpha} = 0 \tag{1.18}$$

It will be seen from equations (1.18) that h is independent of α. But h is a function typical of the boundary value problem, and $h \to 0$ when $\alpha \to \infty$. Hence $h \equiv 0$, and we can rewrite the last three equations of (1.18) as follows

$$\frac{\partial \mathbf{w}^*}{\partial t} = \frac{\partial^2 \mathbf{w}^*}{\partial \alpha^2}, \qquad \operatorname{Div} \mathbf{w}^* + \frac{\partial w_\alpha}{\partial \alpha} = 0 \tag{1.19}$$

Here, $\mathbf{w}^*$ is a two-dimensional vector with components w_ξ and w_η, and Div denotes a two-dimensional divergence operation, computed from the values of the two-dimensional vectors on surface S (in this computation α is considered to be a parameter). The boundary conditions for $\mathbf{w}^*$ is found from (1.17) and $\mathbf{w} = \mathbf{w}^* + O(\nu^{1/2})$, with the accuracy of the order of $\nu^{1/2}$ and is

$$\mathbf{w}^* = \boldsymbol{\omega} \times \mathbf{r} - \mathbf{v}^\circ \qquad \text{for } \alpha = 0$$

It follows from the boundary condition (1.10) that the right-hand side of this equation represent a vector, tangent to the surface S. Vector $\mathbf{w}^*$ must also satisfy the initial zero condition (see (1.16)) and the condition at infinity

$$\mathbf{w}^* = 0 \quad \text{when } t = t_0, \qquad \mathbf{w}^* \to 0 \quad \text{when } \alpha \to \infty$$

The solution of the thermal conductivity equation (1.19), with the stated boundary and initial conditions, has the form [8]

$$\mathbf{w}^* = \frac{\alpha}{2\sqrt{\pi}} \int_{t_0}^{t} \frac{\boldsymbol{\omega}(\tau) \times \mathbf{r} - \mathbf{v}^\circ(\mathbf{r}, \tau)}{(t-\tau)^{3/2}} \exp \frac{-\alpha^2}{4(t-\tau)} \, d\tau \tag{1.20}$$

Substituting (1.20) into the second equation of (1.19) and integrating with respect to α with $w_\alpha \to 0$ when $\alpha \to \infty$, we obtain

$$w_\alpha = \frac{1}{\sqrt{\pi}} \int_{t_0}^{t} \frac{\operatorname{Div}[\omega(\tau) \times \mathbf{r} - \mathbf{v}^\circ(\mathbf{r}, \tau)]}{(t-\tau)^{1/2}} \exp \frac{-\alpha^2}{4(t-\tau)} d\tau \tag{1.21}$$

Reverting to the variables $\zeta = \alpha\nu^{1/2}$, and $w_\zeta = \nu^{1/2} w_\alpha$, we rewrite the equations (1.20) and (1.21) as follows

$$\begin{aligned} \mathbf{w}^*(\xi, \eta, \zeta, t) &= \frac{\zeta}{2\sqrt{\pi\nu}} \int_{t_0}^{t} \frac{[\omega(\tau) \times \mathbf{r} - \mathbf{v}^\circ(\mathbf{r}, \tau)]}{(t-\tau)^{3/2}} \exp \frac{-\zeta^2}{4\nu(t-\tau)} d\tau \\ w_\zeta(\xi, \eta, \zeta, t) &= \frac{\sqrt{\nu}}{\sqrt{\pi}} \int_{t_0}^{t} \frac{\operatorname{Div}[\omega(\tau) \times \mathbf{r} - \mathbf{v}^\circ(\mathbf{r}, \tau)]}{(t-\tau)^{1/2}} \exp \frac{-\zeta^2}{4\nu(t-\tau)} d\tau \end{aligned} \tag{1.22}$$

Here the position vector $\mathbf{r}$ and the inward normal $\mathbf{n}$ originate at the point ξ, η on the surface s i.e. at $\zeta = 0$.

By virtue of (1.22), the projection of (1.17) on the normal $\mathbf{n}$ to the surface S is

$$\frac{\partial \varphi^1}{\partial n} = -\frac{w_\zeta(\xi, \eta, 0, t)}{\sqrt{\nu}} = \frac{1}{\sqrt{\pi}} \int_{t_0}^{t} \frac{\operatorname{Div}[\mathbf{v}^\circ(\mathbf{r}, \tau) - \omega(\tau) \times \mathbf{r}]}{(t-\tau)^{1/2}} d\tau \quad \text{on } S \tag{1.23}$$

In this way we have obtained for the function φ^1 harmonic in D, Neuman's problem with the condition (1.23). The solution of this problem can be expressed in a manner similar to that of problem (1.12) by a harmonic function independent of time.

Let functions Ψ_0 and Ψ_i be the solutions of the following boundary problems

$$\triangle \Psi_0 = 0 \quad \text{in } D, \quad \partial \Psi_0 / \partial n = \operatorname{Div} \mathbf{a} \quad \text{on } S \tag{1.24}$$

$$\triangle \Psi_i = 0 \quad \text{в } D, \quad \partial \Psi_i / \partial n = \operatorname{Div} (\nabla \Phi_i - \mathbf{e}_i \times \mathbf{r}) \text{ on } S \qquad (i=1,2,3)$$

Then it can be written

$$\varphi^1(\mathbf{r}, t) = \frac{1}{\sqrt{\pi}} \int_{t_0}^{t} \frac{\Psi(\mathbf{r}, \tau)}{\sqrt{t-\tau}} d\tau, \quad \Psi = \Psi_0 + \omega_1 \Psi_1 + \omega_2 \Psi_2 + \omega_3 \Psi_3 \tag{1.25}$$

Taking into account Equations $\mathbf{v}^\circ = \mathbf{a} + \nabla \varphi^\circ$ and (1.13), we see that function φ^1 satisfies the Laplace equation and the boundary condition (1.23), and that the initial condition $\mathbf{v}^1 = \nabla \varphi^1 = 0$ for $t = t_0$ is also satisfied.

Thus, the determination of an asymptotic solution for the boundary problem (1.8), (1.3) and (1.4) is reduced to the following. First, for a given cavity D we have to determine the Joukowski potentials Φ_i which must satisfy the boundary problems (1.14) and the functions Ψ_i $(i = 1, 2, 3)$, which, in turn must satisfy the problems (1.24). We note that for any vector field $\mathbf{p}(\mathbf{r})$, defined on a closed surface S, the following equation is valid

$$\oint_S \operatorname{Div} \mathbf{p} \, dS = 0 \tag{1.26}$$

and is easily proved by means of the Gauss – Ostrogradskii theorem for surface vector fields [9]. Therefore Neumann's problems (1.24), as well as those of (1.14) have unique solutions with the accuracy up to the constant term.

If at the moment of time $t = t_0$ we have a flow with vorticity it will be necessary to determine, in addition to Φ_i and Ψ_i, the vector function $\mathbf{a}$ in accordance with conditions (1.11), and the function Ψ_0 in accordance with (1.24). Functions Φ_i and Ψ_i depend on the form of cavity D only, while functions $\mathbf{a}$ and Ψ_0, also on the initial distribution of velocities $\mathbf{u}_0$. We note that all these functions are independent of time.

The asymptotic solution of the problem (1.8), (1.3) and (1.4) is expressed, in accordance with above formulas, by

$$\mathbf{u} = \mathbf{a} + \nabla\varphi^\circ + \nu^{1/2}\nabla\varphi^1 + \mathbf{w}, \qquad q = -\varphi_t^\circ - \nu^{1/2}\varphi_t^1 + C(t) \tag{1.27}$$

where $C(t)$ is an arbitrary function. Functions φ°, φ^1, and $\mathbf{w}$ are defined here by formulas (1.13), (1.25) and (1.22) in which $\mathbf{v}^\circ = \mathbf{a} + \nabla\varphi^\circ$. Outside the boundary layer, formula (1.27) approximates the exact solution of problem (1.8), (1.3) and (1.4) with an error of the order of ν. Within the boundary layer itself the functions (1.27) define the component of the vector $\mathbf{u}$ normal to S with the error of the order of ν, while for the function q and the component of vector $\mathbf{u}$, tangent to S, this error is of the order of $\nu^{1/2}$.

We shall now indicate the time interval in which the solution arrived at is valid. It was assumed that in expansion (1.9) coefficients $\mathbf{v}^0$, $\mathbf{v}^1$, etc. were of the same order of magnitude. Since $|\mathbf{u}_0| \sim \omega L$, $L \sim 1$, and $T \sim 1$, it follows from (1.10), (1.11) and (1.13) that $|\mathbf{v}^\circ|$, $|\mathbf{a}|$, and φ° are of the order of magnitude of ω. We therefore conclude that, by virtue of (1.24) and (1.25), $\Psi \sim \omega$, and consequently $\varphi^1 \sim \omega\sqrt{t - t_0}$. In order to have either $\mathbf{v}^0$ and $\mathbf{v}^1$, or φ° and φ^1, of the same order of magnitude, we must have $t - t_0 \sim 1$. Thus, the obtained solution is generally valid in the interval of time of the order of the period T of oscillation of the body.

Let us assume that two following conditions are fulfilled: (1) at the initial moment the flow is potential, i.e. curl $\mathbf{u}_0 = 0$ and consequently $\mathbf{a} = 0$ and $\Psi_0 = 0$; (2) functions

$$\int_{t_0}^{t} \frac{\omega_i(\tau)\, d\tau}{\sqrt{t-\tau}} \qquad (i = 1, 2, 3)$$

for all values of t $(t_0 \leqslant t < \infty)$ are of the order of magnitude of ω. In this case the motion of the body consists of oscillations about a certain central position, and function φ^1 of (1.25) will be of the order of ω even when $t \to \infty$. In this important case the solution arrived at is valid, with the accuracy given above for all $t \geqslant t_0$. If, however, these conditions are not fulfilled, the motion will be essentially vertical. Such motions in absence of viscosity were considered in [10].

2. Equations of motion of a body with fluid. The equations of motion of a rigid body with a cavity completely filled with fluid can be written as follows

$$m\mathbf{R}_c^{\cdot\cdot} = \mathbf{F}, \quad \mathbf{K}^{\cdot} + m(\mathbf{R}_c - \mathbf{R}_0) \times \mathbf{R}_0^{\cdot\cdot} = \mathbf{m}_0 \tag{2.1}$$

Here, m is the mass of the body with fluid, $\mathbf{R}_c$ and $\mathbf{R}_0$ are the absolute position vectors of the center of inertia C of the system and of the pole O, rigidly connected to the body, $\mathbf{F}$ is the principal vector of all external forces acting on the body, $\mathbf{m}_0$ is the principal moment of these forces about the pole O, and the dot above a symbol indicates a derivative with respect to time. We denote by $\mathbf{K}$ the kinetic moment of the whole system in its motion relative to the pole O

$$\mathbf{K} = \int \mathbf{r} \times \mathbf{u}\, dm = \mathbf{J}_0 \cdot \boldsymbol{\omega} + \rho_0 \int_D \mathbf{r} \times \mathbf{u}\, dV \tag{2.2}$$

Here the first term represents the kinetic moment of the rigid body ($\mathbf{J}_0$ is its tensor of inertia relative to the pole O), and the second is the kinetic moment of the fluid in the cavity.

Equations (2.1) must be supplemented by kinematic relationships (for the direction cosines, or Euler's angles), which in this case are the same as for a rigid body without any fluid. The derivation of equations of motion is reduced, in this manner, to the determination of the kinetic moment of the relative motion of the fluid (second term of (2.2)).

We shall compute this kinetic moment with an error of the order of $\sim \nu$. For this purpose we shall use $\mathbf{u}$ as given in (1.27). In addition to that we have in the region of the boundary layer D_1 $\mathbf{w} = \mathbf{w}^* + O(\nu^{1/2})$. As the volume of D_1 is $O(\nu^{1/2})$, we can assume that in this region $\mathbf{w} = \mathbf{w}^*$. This will give an error of the order of ν when computing $\mathbf{K}$ from (2.2). We assume that outside D_1 $\mathbf{w} = 0$. We then have

$$\mathbf{K} = \mathbf{J}_0 \cdot \boldsymbol{\omega} + \mathbf{K}^\circ + \rho_0 \int_D \mathbf{r} \times \nabla \varphi^\circ\, dV + \rho_0 \sqrt{\nu} \int_D \mathbf{r} \times \nabla \varphi^1\, dV +$$
$$+ \rho_0 \int_{D_1} \mathbf{r} \times \mathbf{w}^*\, dV + O(\nu), \qquad \mathbf{K}^\circ = \rho_0 \int_D \mathbf{r} \times \mathbf{a}\, dV \tag{2.3}$$

It is evident that $\mathbf{K}^0$ is a constant vector. The third term of (2.3) represents the kinetic moment of potential motion of the perfect fluid about the point O. By virtue of (1.13) and (1.14), this term can be expressed by $\mathbf{J} \cdot \boldsymbol{\omega}$, where $\mathbf{J}$ is the tensor of associated masses [1 and 2]. Components of the symmetric tensor $\mathbf{J}$ (associated moments of inertia) are

$$J_{ij} = J_{ji} = \rho_0 \int_D (\mathbf{r} \times \nabla \Phi_i) \cdot \mathbf{e}_j\, dV = -\rho_0 \oint_S (\mathbf{r} \times \mathbf{n}) \cdot \mathbf{e}_j \Phi_i\, dS = -\rho_0 \oint_S \Phi_i \frac{\partial \Phi_j}{\partial n}\, dS \qquad (i, j = 1, 2, 3) \tag{2.4}$$

We reduce the fourth term of (2.3) to a surface integral with the aid of equation $\mathbf{r} \times \nabla \varphi^1 = -\operatorname{curl}(\mathbf{r}\varphi^1)$, and then substitute into it the expression for φ^1 from (1.25)

$$\frac{\rho_0 \sqrt{\nu}}{\sqrt{\pi}} \int_{t_0}^{t} \left[\oint_S (\mathbf{n} \times \mathbf{r})\, \Psi\, dS \right] \frac{d\tau}{\sqrt{t - \tau}}$$

We substitute the expression for $\mathbf{w}^*$ from (1.22) into the fifth term of formula (2.3), and replace the volume integral over D_1 with the surface integral over S and the integral with respect to ζ from 0 to ∞. This is permissible because $\mathbf{w}^*$ rapidly tends to 0 when $\zeta \gg \nu^{1/2}$. After the integration with respect to ζ, this term becomes

$$\frac{\rho_0 \sqrt{\nu}}{\sqrt{\pi}} \int_{t_0}^{t} \left[\oint_S \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r} - \mathbf{v}^\circ)\, dS \right] \frac{d\tau}{\sqrt{t-\tau}} \qquad (\mathbf{v}^\circ = \mathbf{a} + \nabla \varphi^\circ)$$

The kinetic moment (2.3) can now be written in the form

$$\mathbf{K} = \mathbf{I} \cdot \boldsymbol{\omega} + \mathbf{K}^\circ + \mathbf{K}^* \qquad (\mathbf{I} = \mathbf{J}_0 + \mathbf{J}) \tag{2.5}$$

Here $\mathbf{K}^*$ is the sum of the last two terms of Formula (2.3) which by virtue of the above relationships and of Formulas (1.13) and (1.25), equals

$$\mathbf{K}^* = \frac{\rho_0 \sqrt{\nu}}{\sqrt{\pi}} \int_{t_0}^{t} \left\{ \oint_S \left[-\mathbf{r} \times \mathbf{a} - (\mathbf{r} \times \mathbf{n})\, \Psi_0 \right] dS + \right.$$
$$\left. + \sum_{j=1}^{3} \omega_j \oint_S \left[\mathbf{r} \times (\mathbf{e}_j \times \mathbf{r} - \nabla \Phi_j) - (\mathbf{r} \times \mathbf{n})\, \Psi_j \right] dS \right\} \frac{d\tau}{\sqrt{t-\tau}} \tag{2.6}$$

Let us now introduce new notations

$$\mathbf{b} = \rho_0 \frac{\sqrt{\nu}}{\sqrt{\pi}} \oint_S [-\mathbf{r} \times \mathbf{a} - (\mathbf{r} \times \mathbf{n})\, \Psi_0]\, dS$$
$$B_{ij} = \rho_0 \frac{\sqrt{\nu}}{\sqrt{\pi}} \oint_S [\mathbf{r} \times (\mathbf{e}_j \times \mathbf{r} - \nabla \Phi_j) - (\mathbf{r} \times \mathbf{n})\, \Psi_j] \cdot \mathbf{e}_i\, dS \qquad (i, j = 1, 2, 3) \tag{2.7}$$

and rewrite the formula (2.6) as follows

$$\mathbf{K}^* = \int_{t_0}^{t} (\mathbf{b} + \mathbf{B} \cdot \boldsymbol{\omega}) \frac{d\tau}{\sqrt{t-\tau}} = 2\mathbf{b} \sqrt{t - t_0} + \mathbf{B} \cdot \int_{t_0}^{t} \frac{\boldsymbol{\omega}(\tau)\, d\tau}{\sqrt{t-\tau}} \tag{2.8}$$

Here $\mathbf{b}$ is a constant vector, and $\mathbf{B}$ is a constant affine tensor with components B_{ij}.

In accordance with the boundary conditions (1.14) we have

$$\mathbf{r} \times \mathbf{n} = \sum_{i=1}^{3} \mathbf{e}_i \frac{\partial \Phi_i}{\partial n}$$

Substituting this expression into Equations (2.7), applying Green's theorem, and using the boundary conditions (1.24), we obtain

$$\mathbf{b} = \frac{\rho_0 \sqrt{\nu}}{\sqrt{\pi}} \oint_S \left(-\mathbf{r} \times \mathbf{a} - \sum_{i=1}^{3} \mathbf{e}_i \frac{\partial \Phi_i}{\partial n} \Psi_0 \right) dS =$$
$$= \frac{\rho_0 \sqrt{\nu}}{\sqrt{\pi}} \oint_S \left(-\mathbf{r} \times \mathbf{a} - \sum_{i=1}^{3} \mathbf{e}_i \Phi_i \operatorname{Div} \mathbf{a} \right) dS$$
$$B_{ij} = \frac{\rho_0 \sqrt{\nu}}{\sqrt{\pi}} \oint_S \left[\mathbf{e}_i \cdot \mathbf{e}_j r^2 - (\mathbf{e}_i \cdot \mathbf{r})(\mathbf{e}_j \cdot \mathbf{r}) - \mathbf{e}_i \cdot (\mathbf{r} \times \nabla \Phi_j) - \frac{\partial \Phi_i}{\partial n} \Psi_j \right] dS = \tag{2.9}$$
$$= \frac{\rho_0 \sqrt{\nu}}{\sqrt{\pi}} \oint_S \left[\mathbf{e}_i \cdot \mathbf{e}_j r^2 - (\mathbf{e}_i \cdot \mathbf{r})(\mathbf{e}_j \cdot \mathbf{r}) + (\mathbf{r} \times \mathbf{e}_i) \cdot \nabla \Phi_j - \Phi_i \operatorname{Div} (\nabla \Phi_j - \mathbf{e}_j \times \mathbf{r}) \right] dS$$

For any given vector field $\mathbf{q}(\mathbf{r})$ and scalar function $f(\mathbf{r})$ on the surface S the following formula is valid [9]

$$\operatorname{Div} f\mathbf{q} = f \operatorname{Div} \mathbf{q} + \mathbf{q}\cdot \operatorname{Grad} f$$

Here Grad denotes the operation of taking the gradient along the surface S. If the function f is also defined outside S, then the Grad operation is related to the gradient in a three-dimensional space by the expression $\operatorname{Grad} f = \nabla f - \mathbf{n}\,(\partial f / \partial n)$. With these equations and formula (1.26) we find, that for any closed surface S

$$\oint_S f \operatorname{Div} \mathbf{q}\, dS = \oint_S \left(\mathbf{n}\frac{\partial f}{\partial n} - \nabla f\right)\mathbf{q}\, dS \qquad (2.10)$$

We shall use this identity (2.10) for the transformation of integrals (2.9). Taking into account the fact that on surface S $\mathbf{an} = 0$, we obtain for vector $\mathbf{b}$

$$\begin{aligned}\mathbf{b} &= \frac{\rho_0 \sqrt{\nu}}{\sqrt{\pi}} \oint_S \left[-\mathbf{r}\times\mathbf{a} + \sum_{i=1}^{3} \mathbf{e}_i (\nabla\Phi_i\cdot\mathbf{a})\right] dS = \\ &= \frac{\rho_0 \sqrt{\nu}}{\sqrt{\pi}} \sum_{i=1}^{3} \mathbf{e}_i \oint_S (\mathbf{r}\times\mathbf{e}_i + \nabla\Phi_i)\cdot\mathbf{a}\, dS\end{aligned} \qquad (2.11)$$

A similar transformation of the second integral of (2.9) yields

$$\begin{aligned}B_{ij} &= \frac{\rho_0 \sqrt{\nu}}{\sqrt{\pi}} \oint_S \Big[r^2\mathbf{e}_i\cdot\mathbf{e}_j - (\mathbf{r}\cdot\mathbf{e}_i)(\mathbf{r}\cdot\mathbf{e}_j) + (\mathbf{r}\times\mathbf{e}_i)\cdot\nabla\Phi_j - \\ &- \left(\mathbf{n}\frac{\partial\Phi_i}{\partial n} - \nabla\Phi_i\right)(\nabla\Phi_j - \mathbf{e}_j\times\mathbf{r})\Big] dS = \frac{\rho_0 \sqrt{\nu}}{\sqrt{\pi}} \oint_S \Big\{\Big[r^2\mathbf{e}_i\cdot\mathbf{e}_j - (\mathbf{r}\cdot\mathbf{e}_i)(\mathbf{r}\cdot\mathbf{e}_j)\Big] + \\ &+ (\mathbf{r}\times\mathbf{e}_i)\cdot\nabla\Phi_j + (\mathbf{r}\times\mathbf{e}_j)\cdot\nabla\Phi_i + \nabla\Phi_i\cdot\nabla\Phi_j - \frac{\partial\Phi_i}{\partial n}\Big[\mathbf{n}\cdot\nabla\Phi_j - \mathbf{n}\cdot(\mathbf{e}_j\times\mathbf{r})\Big]\Big\} dS\end{aligned}$$

The expression in the second square brackets of the last integral is zero by virtue of (1.14), and the expression in the first bracket can be transformed by using the following known identity from vector algebra [7]

$$r^2(\mathbf{e}_i\cdot\mathbf{e}_j) - (\mathbf{r}\cdot\mathbf{e}_i)(\mathbf{r}\cdot\mathbf{e}_j) = (\mathbf{r}\times\mathbf{e}_i)\cdot(\mathbf{r}\times\mathbf{e}_j)$$

Finally we obtain

$$B_{ij} = \frac{\rho_0 \sqrt{\nu}}{\sqrt{\pi}} \oint_S (\mathbf{r}\times\mathbf{e}_i + \nabla\Phi_i)\cdot(\mathbf{r}\times\mathbf{e}_j + \nabla\Phi_j)\, dS \qquad (2.12)$$

It follows from this that $\mathbf{B}$ is a symmetric tensor ($B_{ij} = B_{ji}$). It can be brought into a diagonal form, i.e. we can obtain $B_{ij} = 0$ for $i \neq j$, by a suitable selection of the coordinate system $ox_1x_2x_3$. Since in any system of coordinates $B_{ii} > 0$, it follows that the eigenvalues of the tensor $\mathbf{B}$ are real and positive. Dependence of the components B_{ij}, as well as of J_{ij} of the tensor (2.4) on the cavity form is expressed by the Joukowski potentials Φ_i. For the computation of tensor $\mathbf{B}$ it is sufficient to determine the functions Φ_i and calculate the integrals (2.12). Vector $\mathbf{b}$ is further deptendent on the vector function $\mathbf{a}$. We note that the computation of $\mathbf{b}$ and $\mathbf{B}$ does not require the solution of the boundary problems (1.24).

Let us now consider the effect of altering the pole O, on the vector **b** and tensor **B**. Let points O' and O be stationary with respect to the rigid body. We denote the vector connecting these two point by $\mathbf{d}(t) = OO'$. We then have, for the relative velocities and position vectors of points of the fluid, the following expressions

$$\mathbf{r}' = \mathbf{r} - \mathbf{d}(t), \qquad \mathbf{u}' = \mathbf{u} - \mathbf{d}^{\cdot}(t), \qquad \mathbf{u}_0' = \mathbf{u}_0 - \mathbf{d}^{\cdot}(t_0)$$

Here $\mathbf{u}_0$ is the initial velocity, and the primes indicate values relative to the pole O'. It follows from (1.11) that in this case $\mathbf{a}' = \mathbf{a}$. The boundary conditions (1.14) for the functions Φ_i' and Φ_i with respect to the poles O' and O are related to each other by

$$\partial\Phi'_i / \partial n = \partial\Phi_i / \partial n - (\mathbf{e}_i \times \mathbf{d})\cdot\mathbf{n} \text{ on } S \ (i = 1, 2, 3)$$

Consequently, we have for Φ_i' and Φ_i, everywhere in D

$$\Phi_i' = \Phi_i - (\mathbf{e}_i \times \mathbf{d})\cdot\mathbf{r}, \qquad \mathbf{r}' \times \mathbf{e}_i + \nabla\Phi_i' = \mathbf{r} \times \mathbf{e}_i + \nabla\Phi_i$$

It follows immediately from these equations and from Formulas (2.11) and (2.12) that $\mathbf{b}' = \mathbf{b}$ and $\mathbf{B}' = \mathbf{B}$, i.e. that vector **b** and tensor **B** are invariant under the change of the pole O. Consequently vector $\mathbf{K}^*$ is also independent of the pole, which can therefore be arbitrarily selected.

We note that this approach is applicable without basic difficulties to multiply-connected cavities. If at $t = t_0$ the flow is potential, as is assumed below, then the analysis of the multiply connected cavities does not differ at all from that of a simply connected cavity.

Thus, the kinetic moment **K** is expressed by the formula (2.5) where $\mathbf{K}^*$ is defined by the formula (2.8), and **b** and **B** by formulas (2.11) and (2.12). The substitution of **K** into Equations (2.1) yields a system of integro-differential equations of motion of a body filled with a fluid at large Reynolds numbers. These equations are generally valid in the interval of time $t - t_0 \sim 1$ with an error of the order of ν. If however, two conditions formulated at the end of Chapter 1 are fulfilled the validity of these equations is extended to $t - t_0 \gg 1$. In this most important case, considered in the following, we have $\mathbf{K}^0 = 0$ and $\mathbf{b} = 0$, while the magnitude of $\mathbf{K}^*$ is, for all $t \geqslant t_0$ of the order of $\nu^{1/2}$, as compared to $\mathbf{I}\cdot\boldsymbol{\omega}$ given by (2.5). This aspect makes it possible to solve the system of equations (2.1) by the method of small parameter.

We note that formulas (2.5) and (2.8) make it possible to derive equations of motion for more complicated systems than a rigid body with a cavity filled with a fluid (for example, a system of bodies with fluid filled cavities).

3. Certain specific cavity forms. We shall now determine tensor **B** for certain specific forms of cavity, and at the same time establish expressions for tensor **J**.

1. Let O_{x_3} be the axis of symmetry of the cavity. Then the Joukowski potentials by virtue of (1.14) have the following properties.

$$\Phi_i(x_1, x_2, x_3) = -\Phi_i(-x_1, -x_2, x_3) \ (i = 1, 2), \qquad \Phi_3(x_1, x_2, x_3) = \Phi_3(-x_1, -x_2, x_3)$$

In this case we obtain from (2.4) and (2.12) $J_{i3} = 0$ and $B_{i3} = 0$ for $i = 1, 2$, i.e. the axis of symmetry of the cavity is the principal axis of the tensors **J** and **B**. If all the three axes Ox_1, Ox_2, and Ox_3 are the axes of symmetry of the cavity, then the tensors **J** and **B** will have a diagonal form on these axes.

2. We shall consider a cavity enclosed by an ellipsoid

$$x_1^2 a_1^{-2} + x_2^2 a_2^{-2} + x_3^2 a_3^{-2} = 1$$

In this case the Joukowski potential Φ_3 is [1]

$$\Phi_3 = [(a_1^2 - a_2^2) / (a_1^2 + a_2^2)] x_1 x_2 \tag{3.1}$$

while Φ_1 and Φ_2 are obtained by cyclic transposition of all indices.

The associated moments of inertia (2.4) are expressed by the known formula [1]

$$J_{33} = {}^4/_{15}\pi\rho_0 a_1 a_2 a_3 (a_1^2 - a_2^2)^2 (a_1^2 + a_2^2)^{-1}, \quad J_{ij} = 0 \text{ when } i \neq j$$

where J_{11} and J_{22} are obtained by cyclic transposition of indices. Substituting (3.1) into (2.12) and introducing into the latter new variables

$$x_1 = a_1 \sin u \cos v, \qquad x_2 = a_2 \sin u \sin v, \qquad x_3 = a_3 \cos u$$

we shall reduce the integral (2.12) to a double integral

$$B_{33} = \frac{16\rho_0 \sqrt{\pi\nu}\, a_1^5 a_2^5}{a_3^2 (a_1^2 + a_2^2)^2} F(\alpha_1, \alpha_2), \qquad \alpha_1 = \frac{a_1}{a_3}, \qquad \alpha_2 = \frac{a_2}{a_3}$$

$$F(\alpha_1, \alpha_2) = \frac{2}{\pi} \int_0^{1/2\pi}\int_0^{1/2\pi} \sin^3 u \left(\frac{\cos^2 v}{\alpha_1^2} + \frac{\sin^2 v}{\alpha_2^2}\right)\left[\cos^2 u + \sin^2 u \left(\frac{\cos^2 v}{\alpha_1^2} + \frac{\sin^2 v}{\alpha_2^2}\right)\right]^{1/2} du\, dv \tag{3.2}$$

We note that $B_{ij} = 0$ for $i \neq j$, and B_{11} and B_{22} are obtained by cyclic transposition of indices.

Function $F(\alpha_1, \alpha_2)$ is dependent on two parameters α_1 and α_2, and decreases monotonely with the increase of either of them. Since $F(\alpha_1, \alpha_2) = F(\alpha_2, \alpha_1)$, it can be assumed that $\alpha_1 \geqslant \alpha_2$.

The integration of (3.2) with respect to u can be carried out in terms of elementary functions with the result

$$F(\alpha_1, \alpha_2) = \frac{2}{\pi}\int_0^{1/2\pi} G(z)\, dv, \qquad z = \frac{\cos^2 v}{\alpha_1^2} + \frac{\sin^2 v}{\alpha_2^2}$$

$$G(z) = \frac{z}{8(z-1)}\left[3z - 2 + \frac{z(3z-4)}{\sqrt{1-z}} \ln \frac{1+\sqrt{1-z}}{\sqrt{z}}\right] \quad \text{when } z \leqslant 1 \tag{3.3}$$

$$G(z) = \frac{z}{8(z-1)}\left[3z - 2 + \frac{z(3z-4)}{\sqrt{z-1}} \sin^{-1}\left(\frac{z-1}{z}\right)^{1/2}\right] \text{when } z \geqslant 1$$

When $z \geqslant 0$ the function $G(z)$ is continuous; when $z \to 0$ we have $G(z) \sim \frac{1}{4} z$, $G(1) = 2/3$, and when $z \to \infty$ we have $G(z) \to {}^3/_{16}\pi z^{3/2}$. These formulas make it possible to determine function $F(\alpha_1, \alpha_2)$ when certain relationships between the parameters α_1 and α_2 exist

$$F(\alpha, \alpha) = G(\alpha^{-2}), \qquad F(1, 1) = {}^2/_3$$

$$F(\alpha_1, \alpha_2) = {}^1/_4 \alpha_2^{-3} \qquad \text{for } \alpha_1 \geqslant \text{const} > 0,\ \alpha_2 \to 0$$

$$F(\alpha_1, \alpha_2) = {}^1/_8 (\alpha_1^{-2} + \alpha_2^{-2}) \qquad \text{for } \alpha_1, \alpha_2 \to \infty$$

We shall analyse with the aid of these formulas certain particular forms of ellipsoids.

For the ellipsoid of revolution we have $B_{33} = 4\rho_0 \sqrt{\pi\nu}\, a^6 a_3^{-2} G(a_3^2 a^{-2}) \qquad (a = a_1 = a_2)$

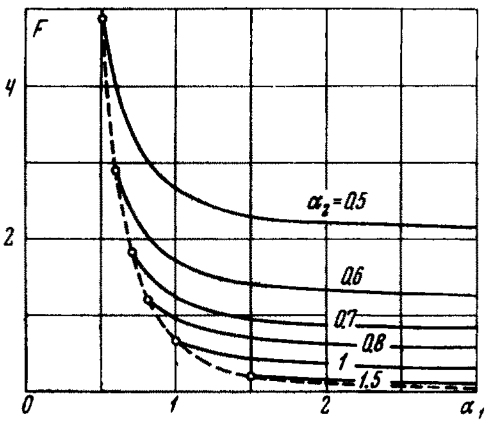

FIG. 2

where the function $G(z)$ is defined by (3.3). Using the assymptotics of function $G(z)$ we find, for a strongly elongated ellipsoid of revolution

$$B_{33} = {}^3/_4 \pi\rho_0 \sqrt{\pi\nu}\, a^3 a_3$$

For a strongly flattened ellipsoid $(a_3 \ll a_1,\ a_3 \ll a_2)$ we have

$$B_{11} = B_{22} = 4\rho_0 \sqrt{\pi\nu}\, a_1 a_2 a_3^2$$
$$B_{33} = 2\rho_0 \sqrt{\pi\nu}\, a_1^3 a_2^3 / (a_1^2 + a_2^2)$$

These formulas are easily derived by cyclic transposition of indices in Formulas (3.2), and the use of the asymptotic properties of the function F. In particular, for a strongly flattened ellipsoid of revolution $(a_1 = a_2 = a,\ a_3 \ll a)$ we have

$$B_{11} = B_{22} = 4\rho_0 \sqrt{\pi\nu}\, a^2 a_3^2$$
$$B_{33} = \rho_0 \sqrt{\pi\nu}\, a^4$$

Finally, in the case of a spherical cavity of radius a we find directly from (3.2)

$$B_{ii} = {}^8/_3 \rho_0 \sqrt{\pi\nu}\, a^4 \qquad (i = 1, 2, 3)$$

In the general case $(a_1 \neq a_2)$, function $F(\alpha_1, \alpha_2)$ was determined by calculating the quadratures of (3.3) on an electronic computer. Values of the function F, calculated for several values of α_1 and α_2 are given in Table 1. Function $F(\alpha_1, \alpha_2)$ is represented diagrammatically on Fig. 2 for fixed values of α_2 with $\alpha_1 \geqslant \alpha_2$, with the values of α_2 given at each line, and the broken line representing $F(\alpha_1, \alpha_1)$.

TABLE 1

α_2 \ α_1	0.1	0.2	0.5	1	2	5	10
0.1	590.0	311.8	258.7	252.5	251.1	250.7	250.6
0.2	—	74.11	36.16	32.57	31.79	31.59	31.56
0.5	—	—	4.891	2.621	2.218	2.124	2.111
1	—	—	—	0.6667	0.3665	0.3047	0.2970
2	—	—	—	—	0.1036	0.05499	0.04944
5	—	—	—	—	—	0.01168	0.007068
10	—	—	—	—	—	—	0.002638

3. Let the cavity D have the shape of a solid revolution. Let the axis Ox_3 be the axis of rotation, and let us introduce cylindrical coordinates ρ, φ, and z

$$x_1 = \rho\cos\varphi, \qquad x_2 = \rho\sin\varphi, \qquad x_3 = z$$

In this system of coordinates the boundary conditions (1.14) will have the form

$$\partial\Phi_1 / \partial n = -\sin\varphi\,(zn_\rho - \rho n_z), \qquad \partial\Phi_2 / \partial n = \cos\varphi\,(zn_\rho - \rho n_z), \qquad \partial\Phi_3 / \partial n = 0 \tag{3.4}$$

where n_ρ and n_z are the projections of the unit vector $\mathbf{n}$ on the ρ and z axes respectively. We note that in this case the normal $\mathbf{n}$ lies in the meridian plane $\varphi = \text{const}$. It will be seen from Equations (3.4) that the Joukowski potentials can be sought in the form of

$$\Phi_1 = -\sin\varphi\, f(\rho, z), \qquad \Phi_2 = \cos\varphi\, f(\rho, z), \qquad \Phi_3 = 0 \tag{3.5}$$

The boundary problem for the function $f(\rho, z)$ is obtained from (1.14) by taking into account (3.4) as follows

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right) + \frac{\partial^2 f}{\partial z^2} - \frac{f}{\rho^2} = 0 \quad \text{in } \Delta, \qquad n_\rho\left(\frac{\partial f}{\partial\rho} - z\right) + n_z\left(\frac{\partial f}{\partial z} + \rho\right) = 0 \text{ on } \Gamma \tag{3.6}$$

Here the two-dimensional region Δ is the intersection of the cavity with the half-plane $\varphi = \text{const}$, and Γ is the boundary of this region. If the axis of rotation intersects the cavity D, then the boundary of region Δ contains segments of the axis $\rho = 0$. In this case these segments are not included in the countour Γ, and the condition of boundedness of function f is substituted on them for the boundary condition (3.6). It is not difficult to compute the expressions of integrals (2.12) using the formulas (3.5)

$$\begin{aligned} \mathbf{r}\times\mathbf{e}_1 + \nabla\Phi_1 &= \mathbf{e}_1 \sin\varphi\cos\varphi\,(-f_\rho + \rho^{-1}f) + \mathbf{e}_2\,(z - \sin^2\varphi f_\rho - \\ &\quad - \cos^2\varphi\,\rho^{-1}f) - \mathbf{e}_3 \sin\varphi\,(\rho + f_z) \\ \mathbf{r}\times\mathbf{e}_3 + \nabla\Phi_3 &= \rho\,(\mathbf{e}_1\sin\varphi - \mathbf{e}_2\cos\varphi) \end{aligned} \tag{3.7}$$

A similar formula is obtained for $\mathbf{r}\times\mathbf{e}_2 + \nabla\Phi_2$. We substitute (3.7) into (2.12) and replace the integration over the surface of revolution S with the integration with respect to angle φ (from 0 to 2π) and along the curve Γ. After elementary integration with respect to φ and some simple transformations we obtain

$$\begin{gathered} B_{11} = B_{22} = \rho_0 \sqrt{\pi\nu}\int_\Gamma \left[f_\rho^{\,2} + \rho^{-2}f^2 + 2z\,(z - f_\rho - \rho^{-1}f) + (\rho + f_z)^2\right]\rho\, dl \\ B_{33} = 2\rho_0\sqrt{\pi\nu}\int_\Gamma \rho^3\, dl, \qquad B_{ij} = 0 \qquad (i \neq j) \end{gathered} \tag{3.8}$$

where the indices ρ and z denote partial derivatives.

We note that the component B_{33} is independent of f. It differs from the moment of inertia of the surface S with respect to the axis of rotation Ox_3 by a constant factor only, and can be easily calculated for various cavities of the form of solids of revolution. It is possible, in particular, to compute from (3.8) the component B_{33} for an ellipsoid of revolution and for a sphere.

Computation of B_{11} and B_{22} requires the prior solution of the boundary problem (3.6).

We note that, if we assume

$$f(\rho, z) = \rho z + f_1(\rho, z) \tag{3.9}$$

then the first of the integrals (3.8) can be simplified and reduced to

$$B_{11} = B_{22} = \rho_0 \sqrt{\pi\nu} \int_\Gamma [f_{1\rho}^2 + \rho^{-2} f_1^2 + (2\rho + f_{1z})^2] \rho \, dl \tag{3.10}$$

It follows from (3.6) and (3.9) that the unknown function f_1 satisfies the boundary condition

$$\frac{1}{\rho} \frac{\partial}{\partial\rho}\left(\rho \frac{\partial f_1}{\partial\rho}\right) + \frac{\partial^2 f_1}{\partial z^2} - \frac{f_1}{\rho^2} = 0 \quad \text{in } \Delta, \; n_\rho f_{1\rho} + n_z (2\rho + f_{1z}) = 0 \quad \text{on } \Gamma \tag{3.11}$$

Let us assume that the rigid body, with its cavity of the form of a solid of revolution, is moving in a plane, and that the axis of the cavity executes a translatory motion (this particular case was considered in [4]). In this case only one component of the tensor **B**, namely B_{33} will enter the equations of motion of a body with fluid.

It is also possible to change the integrals (2.4) to cylindrical coordinates and substitute into these the expressions for Φ_i from (3.5). A simple transformation yields integrals similar to those of (3.8)

$$J_{11} = J_{22} = \pi\rho_0 \int_\Gamma (\rho n_z - z n_\rho) f \rho \, dl, \qquad J_{33} = J_{ij} = 0 \qquad (i \neq j;\; i, j = 1, 2, 3) \tag{3.12}$$

It is not difficult to verify that Expression (3.12) is equivalent to the formula arrived at in a different manner on page 86 of [1].

4. We shall consider a cylindrical cavity of radius a and height h. We select the origin of the system of coordinates at the center of symmetry of the cylinder, and make Ox_3 the axis of rotation. The shape of the region Δ of the plane ρz is, in this case, a rectangle the $0 \leqslant \rho \leqslant a$, and $|z| \leqslant h$, while the curve Γ consists of segments $\rho = a$, $|z| \leqslant h$ and $z = \pm h$, $0 \leqslant \rho \leqslant a$. In accordance with (3.8) we have

$$B_{33} = \rho_0 \sqrt{\pi\nu}\, a^3 (4h + a) \tag{3.13}$$

Function $f_1(\rho, z)$ satisfies Equation (3.11) within the rectangle Δ. On its periphery, the boundary conditions of (3.11) apply together with those of boundedness of function f_1 on the axis, which are

$$\frac{\partial f_1}{\partial z} = -2\rho \quad \text{when} \quad |z| = h, \quad \frac{\partial f_1}{\partial \rho} = 0 \quad \text{when } \rho = a, \quad |f_1| < \infty \text{ when } \rho = 0 \tag{3.14}$$

We shall seek the solution of Equation (3.11) with boundary conditions (3.14) in the form given in [1 and 11]

$$f_1(\rho, z) = \sum_{n=1}^{\infty} c_n \sinh(\zeta_n z / a) J_1(\zeta_n \rho / a) \tag{3.15}$$

Here J_1 is the Bessel function, and c_n and ζ_n are constants. It is not difficult to prove that for any values of c_n and ζ_n, function f_1 of (3.15) satisfies Equation (3.11) and the condition of boundedness for $\rho = 0$.

The boundary conditions (3.14) will be satisfied by a solution of (3.15) for $\rho = a$, if the consecutive positive roots of Bessel's function are taken for ζ_n

$$J_1'(\zeta_n) = 0, \qquad (n = 1, 2, \ldots; 0 < \zeta_1 < \zeta_2 < \ldots) \tag{3.16}$$

In order to satisfy the boundary condition (3.14) for $|z| = h$ the coefficients c_n are defined as follows [1 and 11]

$$c_n = -\frac{4a^2}{\zeta_n(\zeta_n^2 - 1)\,\mathrm{ch}\,(\zeta_n h / a)\,J_1(\zeta_n)} \tag{3.17}$$

In the case of a circular cylinder the integral (3.12) becomes

$$J_{11} = J_{22} = 2\pi\rho_0 a \int_0^h z f(a, z)\,dz - 2\pi\rho_0 \int_0^a \rho^2 f(\rho, h)\,d\rho$$

We substitute into this formula $f(\rho, z)$ from (3.9) and f_1 from (3.15), and calculate the integrals, taking into account the formulas (3.16) and (1.17), and the properties of Bessel's function. After the transformations utilising the identities [11]

$$\sum_{n=1}^{\infty} \frac{1}{\zeta_n^2 - 1} = \frac{1}{2}, \qquad \sum_{n=1}^{\infty} \frac{1}{\zeta_n^2(\zeta_n^2 - 1)} = \frac{1}{8} \tag{3.18}$$

we obtain the known result given in [1]

$$J_{11} = J_{22} = \pi\rho_0 a^5 \alpha(H), \quad H = \frac{h}{a}, \quad \alpha(H) = \frac{2}{3}H^3 - \frac{3}{2}H + 16\sum_{n=1}^{\infty} \frac{\tanh(\zeta_n H)}{\zeta_n^3(\zeta_n^2 - 1)} \tag{3.19}$$

We shall now write down the expression for the integral (3.10) for the case of the circular cylinder, taking into account the boundary conditions (3.14)

$$B_{11} = B_{22} = 2\rho_0 \sqrt{\pi\nu} \left\{ \int_0^a (f_{1\rho}^2 + \rho^{-2} f_1^2)_{z=h}\,\rho\,d\rho + a\int_0^h [a^{-2} f_1^2 + (2a + f_{1z})^2]_{\rho=a}\,dz \right\}$$

After integration by parts of the first term of the first integral, and transformations utilising equation (3.11), this expression becomes

$$B_{11} = B_{22} = 2\rho_0 \sqrt{\pi\nu} \left[\int_0^a (f_1 f_{1zz})_{z=h}\,\rho\,d\rho + 4a^3 h + 4a^2 f_1(a, h) + a\int_0^h (a^{-2} f_1^2 + f_{1z}^2)_{\rho=a}\,dz \right]$$

We now substitute into the above f_1 from the formula (3.15), and integrate each term of the obtained series. After the transformations which take into account Formulas (3.17) and (3.18) we finally arrive at

$$B_{11} = B_{22} = 16\rho_0 \sqrt{\pi\nu}\,a^4 \beta(H), \qquad H = h / a$$

$$\beta(H) = \frac{5}{8}H + (1 - H)\sum_{n=1}^{\infty} \frac{\tanh^2(\zeta_n H)}{\zeta_n^2(\zeta_n^2 - 1)} - \sum_{n=1}^{\infty} \frac{(2\zeta_n^4 - 3\zeta_n^2 - 1)\tanh(\zeta_n H)}{\zeta_n^3(\zeta_n^2 - 1)^2} + \\ + 4\sum_{n=2}^{\infty}\sum_{m=1}^{n-1} \frac{1}{(\zeta_n^2 - \zeta_m^2)} \left[\frac{\tanh(\zeta_n H)}{\zeta_n(\zeta_m^2 - 1)} - \frac{\tanh(\zeta_m H)}{\zeta_m(\zeta_n^2 - 1)} \right] \tag{3.20}$$

Series (3.19) and (3.20) are convergent for any $H \geqslant 0$. Numerical values of function $\beta(H)$ were obtained by summation of series (3.20) on a computer. The roots ζ_n of Equation (3.16) were obtained from the tables given in [12] for $n \leqslant 40$, while for $n > 40$ they were computed by means of the known asymptotic formula [13]. Function $\alpha(H)$ was concurrently calculated from (3.19). Some results are given in Table 2, and presented diagrammatically on Fig. 3

TABLE 2

H	0.1	0.2	0.5	1	2	5	10
$\alpha(H)$	0.04817	0.08685	0.1165	0.1909	3.409	76.91	652.7
$\beta(H)$	0.004886	0.01889	0.1011	0.3049	0.7892	2.288	4.788

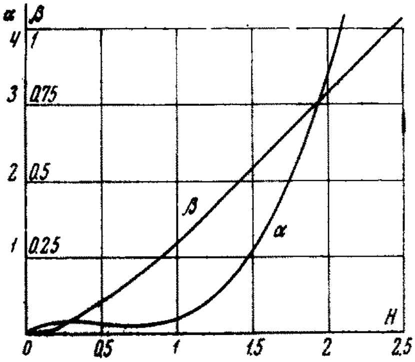

FIG. 3

The assymptotic values of the functions $\alpha(H)$ and $\beta(H)$ are found without difficulties by using identities (3.18)

$$\alpha \sim {}^1/_2 H, \quad \beta \sim {}^1/_2 \; H^2 \quad \text{when} \quad H \to 0$$

$$\alpha \sim {}^2/_3 H^3 - {}^3/_2 H + \alpha_0$$

$$\beta \sim {}^1/_2 H - \beta_0 \quad \text{when} \quad H \to \infty$$

$$\alpha_0 = 1.077, \qquad \beta_0 = 0.2115$$

Constants α_0 and β_0 were determined by comparing them with the numerical solution. These formulas make it possible to write down approximate expressions for the components of tensors **J** and **B** for a cylinder, when $h \ll a$ and $h \gg a$

$$J_{11} = J_{22} = {}^1/_2 \pi \, \rho_0 a^4 h$$

$$B_{11} = B_{22} = 8\rho_0 \, \sqrt{\pi\nu} \, a^2 h^2 \qquad (h \ll a)$$

$$J_{11} = J_{22} = \pi \rho_0 a^2 \, ({}^2/_3 h^3 - {}^3/_2 a^2 h + \alpha_0 a^3)$$

$$B_{11} = B_{22} = 8\rho_0 \sqrt{\pi\nu} \, a^3 \, (h - 2\beta_0 a) \qquad (h \gg a)$$

The latter formulas yield a high degree of accuracy already for $h > 2a$. Component B_{33} is, in all cases, determined from the formula (3.13), while $J_{33} = 0$.

The Joukowski potentials are known for a number of cavity forms [1 and 11]. The computation of tensor **B** for such forms is reduced to the computation of integrals (2.12). Generally, a prior solution of the boundary problems (3.14) is required.

4. Forced oscillations of fluid in the cavity. In par. 1 the motion of fluid in a cavity with the initial condition stipulated by (1.4) was considered. We shall now analyse forced oscillations, assuming that

$$\omega = \Omega e^{pt} \tag{4.1}$$

where Ω is a constant vector, and p is a complex number.

We shall seek the solution of the boundary problem (1.8) and (1.3) in a form in which, for $\nu \ll 1$, the dependence of time is expressed by exp (pt). Similar solutions were analysed in [3 and 4].

As before, the asymptotic solution will be sought in the form given by (1.9). It will be constructed in a form which, for $\nu \to 0$, becomes a potential flow of a perfect fluid. It is necessary to assume in this case that $\mathbf{a} = 0$ and $\mathbf{v}^\circ = \nabla\varphi^\circ$, where φ° is defined by the previous formula (1.13), while other notations are the same as in par. 1. Functions $\mathbf{w}^*$ and w_α are defined by the previously established equations (1.19), and conform to boundary conditions as follows

$$\mathbf{w}^* = \boldsymbol{\omega} \times \mathbf{r} - \nabla\varphi^\circ \quad \text{when } \alpha = 0, \qquad \mathbf{w}^*, \quad w_\alpha \to 0 \quad \text{when } \alpha \to \infty$$

The solution of Equations (1.19) which satisfies the above boundary conditions, and the dependence of which on time is expressed by $\exp(pt)$, has the form

$$\mathbf{w}^* = (\boldsymbol{\omega} \times \mathbf{r} - \nabla\varphi^\circ)\, e^{\sqrt{p}\alpha}, \qquad w_\alpha = \mathrm{Div}\,(\nabla\varphi^\circ - \boldsymbol{\omega} \times \mathbf{r})\, e^{\sqrt{p}\alpha} / \sqrt{p}$$

We choose here the value of $\sqrt{p}$ for which $\mathrm{Re}\sqrt{p} < 0$.

Changing the variables to ζ and w_ζ we obtain, similarily to (1.22)

$$\begin{aligned} \mathbf{w}^*(\xi, \eta, \zeta, t) &= [\boldsymbol{\omega}(t) \times \mathbf{r} - \nabla\varphi^\circ(\mathbf{r}, t)] \exp(\sqrt{p/\nu}\,\zeta)\ (\mathbf{r} = \mathbf{r}(\xi, \eta, 0)) \\ w_\zeta(\xi, \eta, \zeta, t) &= \sqrt{\nu / p}\, \mathrm{Div}\,[\nabla\varphi^\circ(\mathbf{r}, t) - \boldsymbol{\omega}(t) \times \mathbf{r}] \exp(\sqrt{p/\nu}\,\zeta) \end{aligned} \qquad (4.2)$$

We note that the dependence of the functions $\boldsymbol{\omega}$, φ°, $\mathbf{w}^*$, and w_ζ on time is expressed by $\exp(pt)$. For the function φ^1 harmonic in D we have, by virtue of (1.23) and (4.2), the following boundary condition

$$\frac{\partial\varphi^1}{\partial n} = -\frac{w_\zeta(\xi, \eta, 0, t)}{\sqrt{\nu}} = \frac{\mathrm{Div}\,[\boldsymbol{\omega}(t) \times \mathbf{r} - \nabla\varphi^\circ(\mathbf{r}, t)]}{\sqrt{p}} \quad \text{on } S$$

Hence, using (1.24), we can write

$$\varphi^1 = -(\omega_1\Psi_1 + \omega_2\Psi_2 + \omega_3\Psi_3) / \sqrt{p} \qquad (4.3)$$

Thus the required asymptotic solution is expressed by equations similar to those of (1.27)

$$\mathbf{u} = \nabla\varphi^\circ + \nu^{1/2}\nabla\varphi^1 + \mathbf{w}, \qquad q = -\varphi_t^\circ - \nu^{1/2}\varphi_t^1 + C(t)$$

where φ°, $\mathbf{w}$, and φ^1 are defined by the formulas (1.13), (4.2) and (4.3).

The kinetic moment $\mathbf{K}$ (2.2) is computed in the same manner as in par. 2. We have, as in formulas (2.5) and (2.8)

$$\mathbf{K} = \mathbf{I}\cdot\boldsymbol{\omega} + \mathbf{K}^*, \qquad \mathbf{K}^* = -\sqrt{\pi / p}\, \mathbf{B}\cdot\boldsymbol{\omega} \qquad (4.4)$$

5. Oscillations of a body containing a fluid. We shall now deal with the analysis of oscillations of a body containing a fluid, on the assumption that the initial motion of the fluid is potential ($\mathbf{a} = 0$ and $\mathbf{b} = 0$). We shall further assume that either the motion of the point O is uniform and rectilinear (a particular case of this is when the point O is stationary), or that the point O is the center of inertia of the system. The equation of moments (2.1) can then be written, taking into account (2.5), as of (2.5) , as follows

$$\mathbf{I}\cdot\boldsymbol{\omega}^{\cdot} + \mathbf{K}^{*\cdot} = \mathbf{m}_0 \qquad (5.1)$$

We assume that the rigid body with the fluid represents, in the absence of viscosity effects ($\mathbf{K}^* = 0$), a conservative system which has a position of equilibrium, and that the potential energy of this system in the neighbourhood of its position of equilibrium is expressed, with an approximation to the infinitesimals of higher order, by a homogeneous, positive definite quadratic form of its deflections from the position of equilibrium. In this case, as we know, the equilibrium is stable. We shall limit our analysis to small (linear) oscillations in the proximity of the equilibrium position.

We can introduce on these assumptions for a conservative system ($\mathbf{K}^* = 0$) a system of generalised normal coordinates q_1, q_2 and q_3. Vector $\boldsymbol{\omega}$, in the case of linear oscillations, is related to the derivatives of the normal coordinates by

$$\boldsymbol{\omega} = \mathbf{A} \cdot \mathbf{q}^{\cdot} \tag{5.2}$$

where $\mathbf{A}$ is a certain nondegenerate constant matrix, and $\mathbf{q}$ is a vector with the components q_1, q_2 and q_3. The quadratic form of kinetic energy $^1/_2\,(\mathbf{I}\cdot\boldsymbol{\omega})\cdot\boldsymbol{\omega}$ is represented in normal coordinates by $^1/_2(\mathbf{q}^{\cdot}\cdot\mathbf{q}^{\cdot})$. [14].

From this, by virtue of Equation (5.2), we obtain the matrix equation

$$\mathbf{A}'\mathbf{I}\mathbf{A} = \mathbf{E} \tag{5.3}$$

where the prime denotes a transposed matrix, $\mathbf{E}$ is a 3×3 unit matrix.

For a conservative system ($\mathbf{K}^* = 0$) the linearised equations of motion are expressed in normal coordinates by

$$\mathbf{q}^{\cdot\cdot} + \mathbf{D}\cdot\mathbf{q} = 0 \tag{5.4}$$

Here $\mathbf{D}$ is a diagonal matrix with λ_j^2 as its diagonal elements where $\lambda_j > 0$ represent the natural frequencies of oscillation of the conservative system ($j = 1, 2, 3$). Moment of external forces with $\mathbf{K}^* = 0$ is obtained from Equations (5.1), (5.2) and (5.4)

$$\mathbf{m}_0 = \mathbf{I}\cdot\boldsymbol{\omega}^{\cdot} = \mathbf{I}\mathbf{A}\cdot\mathbf{q}^{\cdot\cdot} = -\,\mathbf{I}\mathbf{A}\mathbf{D}\cdot\mathbf{q} \tag{5.5}$$

We assume in accordance with (5.5) that the moment $\mathbf{m}_0$ depends only on coordinates $\mathbf{q}$ also when $\mathbf{K}^* \neq 0$. The equation of motion (5.1) by virtue of (5.2) and (5.5), then becomes

$$\mathbf{I}\mathbf{A}\cdot\mathbf{q}^{\cdot\cdot} + \mathbf{K}^{*\cdot} + \mathbf{I}\mathbf{A}\mathbf{D}\cdot\mathbf{q} = 0$$

Using equation (5.3), we multiply this equation by $\mathbf{A}'$ to obtain

$$\mathbf{q}^{\cdot\cdot} + \mathbf{A}'\cdot\mathbf{K}^{*\cdot} + \mathbf{D}\cdot\mathbf{q} = 0 \tag{5.6}$$

Equation (5.6) will be transformed into a system of integro-differential equations, if we substitute into it the expression for $\mathbf{K}^*$ from (2.8) which, by virtue of Equations (5.2) and $\mathbf{b} = 0$, becomes

$$\mathbf{K}^* = \mathbf{B}\mathbf{A}\cdot\int_{t_0}^{t}\frac{\mathbf{q}^{\cdot}(\tau)\,d\tau}{\sqrt{t-\tau}}$$

As the components of tensor $\mathbf{B}$ are proportional to the small parameter $\sqrt{\nu}$ (see (2.12)), the second term of (5.6) represents a small perturbation. The solution of the system of

Equations (5.6), or (2.1) can be obtained either by the method of small parameters, or numerically. We shall consider here one important class of particular solutions.

The most interesting cases are those of perturbations in the natural oscillations of a system induced by viscosity effects. If an unperturbed system ($\mathbf{K}^* = 0$) is in its j-th mode of characteristic oscillations, all coordinates, with the exception of q_j, are equal to zero. We shall consider the motion of a perturbed system oscillating close to the j-th characteristic oscillation of such a system without perturbation. We can, therefore, assume that all coordinates, with the exception of q_j are small. In the first approximation with respect to parameter $\sqrt{\nu}$ it will be sufficient to retain the terms dependent of q_j only in the perturbing term of the j-th equation of the system (5.6). We shall seek the expression for $q_j(t)$ in the form

$$q_j(t) = C_j e^{p_j}$$

where C_j and p_j are constants. Then, taking for $\mathbf{K}^*$ the expression from (4.4), and with regard to formula (5.2), we obtain the j-th equation of (5.6) in the form

$$C_j p_j^2 e^{p_j t} - \sqrt{\pi / p_j}\, C_j p_j^2 (\mathbf{A}'\mathbf{B}\mathbf{A})_{jj} e^{p_j t} + \lambda_j^2 C_j e^{p_j t} = 0$$

We assume here that $\operatorname{Re} \sqrt{p_j} < 0$, while the index jj indicates the j-th diagonal element of the matrix $\mathbf{A}'\mathbf{B}\mathbf{A}$. From this we derive the characteristic equation for the determination of exponent p_j

$$p_j^2 + \lambda_j^2 = \sqrt{\pi / p_j}\, p_j^2 (\mathbf{A}'\mathbf{B}\mathbf{A})_{jj} \tag{5.7}$$

Next we shall find the roots of Equation (5.7), approximating the roots $\pm i\lambda_j$ of the characteristic equation of the unperturbed system. Let us substitute into (5.7)

$$p_j = \pm i\lambda_j + \delta_j$$

where δ_j is an infinitesimal of the order of magnitude of the components of the tensor $\mathbf{B}$. From (5.7) we obtain the first approximation of δ_j

$$\delta_j = \frac{\sqrt{\pi}\,(\pm i\lambda_j)(\mathbf{A}'\mathbf{B}\mathbf{A})_{jj}}{2\sqrt{\pm i\lambda_j}} \tag{5.8}$$

Selecting in (5.8) the branch of the root for which $\operatorname{Re} \sqrt{\pm i\lambda_j} < 0$, we finally obtain

$$p_j = \pm i\lambda_j + \delta_j = \pm i\lambda_j - \frac{\sqrt{\pi\lambda_j}\,(1 \pm i)(\mathbf{A}'\mathbf{B}\mathbf{A})_{jj}}{2\sqrt{2}} \tag{5.9}$$

Matrix of the tensor $\mathbf{B}$ is symmetric, as was shown in par. 2, and defines a positive definite quadratic form. But the matrix $\mathbf{A}'\mathbf{B}\mathbf{A}$ has the same properties. Therefore, all diagonal elements of matrix $\mathbf{A}'\mathbf{B}\mathbf{A}$ are positive. Hence, it follows from the formula (5.9) that the presence of viscosity has the effect of introducing into characteristic oscillations a damping decrement ($\operatorname{Re} p_j < 0$), and of decreasing the characteristic frequency by the amount equal to that decrement.

We shall consider a simple example. Let the oscillations of the body be restricted to plane oscillations about a fixed axis which we select as the Ox_3 axis. The kinetic energy of this system, in the absence of viscosity, is $^1/_2\, I_{33}\omega^2$, where I_{33} is the moment of inertia of the body, with its cavity filled with a perfect fluid, about the axis Ox_3. The transition to normal coordinates is effected by means of the scalar function $\omega = q^{\cdot} / \sqrt{I_{33}}$, similar to (5.2). Instead of formula (5.9), we then have

$$p = \pm i\lambda - [\sqrt{\pi\lambda}\,(1 \pm i)B_{33}] / (2\sqrt{2}\,I_{33}) \qquad (5.10)$$

where λ is the characteristic frequency of oscillation of the body in the absence of viscosity, and B_{33} is the component of tensor **B** corresponding to axis Ox_3.

If the oscillations are due to the force of gravity (the body in this case is a pendulum with a cavity completely filled with a fluid at large Reynolds numbers, then

$$\lambda = \sqrt{mgl / I_{33}} \qquad (5.11)$$

where m is the total mass of the body and the fluid, g is the acceleration due to gravity, and l the distance of the center of inertia from the axis of suspension Ox_3.

Plane oscillations of a pendulum with its cavity filled with fluid were analysed in [4], where the cavity was assumed to be a body of revolution about an axis parallel to the axis of suspension. For the purpose of computation of the moment of inertia I_{33}, it can be assumed in this case that the mass of fluid is concentrated on the axis of the cavity,

$$I_{33} = I_0 + m' l'^2 \qquad (5.12)$$

where I_0 is the moment of inertia of the rigid body relative to the axis Ox_3, m' is the mass of the fluid, and l' the distance of the axis of the cavity from the axis Ox_3. A substitution into Formula (5.10) of the expressions for λ from (5.11), for I_{33} from (5.12), and for B_{33} from (3.8), yields the result obtained in [4].

BIBLIOGRAPHY

1. Joukowski N.E. O dvizhenii tverdogo tela, imeiushchego polosti, napolnennymi odnorodnolu kapelnoiu zhidkost'iu. (On the motion of a rigid body with cavities filled with homogeneous dropping fluid). Izbr. soch., Vol.1, Gostekhizdat, 1948.

2. Rumiantsev, V.V. Metody Liapunova v izzledovanii ustoichivosti dvizhenia tverdykh tel s polostiami, napolnennymi zhidkost'iu. (Liapunov's method of analysis of the stability of motion of rigid bodies with cavities filled with fluid). Izv. AN SSSR, Mekhanika i Mashinostroenie, No. 6, 1963.

3. Moiseev, N.N. O kraevykh zadachakh dlia linearizovannykh uravnenii Nav'e-Stoksa v sluchae, kogda viazkost mala. (On the boundary problems for linearised Navier-Stokes equations for cases of low viscosity). Zh. vychisl. matem. i matem. fiz., Vol. 1, No. 3, 1961.

4. Krasnoshchekov, P.S. O kolebaniakh fizicheskogo maiatnika, imeiushchego polosti, zapolnennye viazkoi zhidkost'iu. (On the oscillations of a physical pendulum with cavities filled with a viscous fluid). *PMM*, Vol. 27, No. 2, 1963.

5. Chernous'ko, F.L. Dvizhenie tverdogo tela c polostiami, zapolnennymi viazkoi zhidkost'iu pri malykh chislakh Reynoldsa. (Motion of a rigid body with cavities filled with a viscous fluid at small Reynolds numbers). Zh. vychisl. matem. i matem. fiz., Vol. 5, No. 6, 1965.

6. Vishik, M.I. and Liusternik L.A. Reguliarnoie vyrozhdenie i pogranichnyi sloi dlia lineinykh differentsialnykh uravnenii s malym parametrom. (Regular degeneration and boundary layer for linear differential equations with a small parameter). Uspekhi matem. nauk, Vol. 12, No. 5, (77), 1957.

7. Kochin, N.E. Vectornoe ischislenie i nachala tenzornogo ischislenia. (Vector Analysis and Introduction to Tensor Analysis). Izdvo. AN SSSR, 1961.

8. Sneddon, I.N. Preobrazovaniia Fourie. (Fourier Transformations). Izd. inostr. lit., 1955.

9. Norden, A.P. Teoriia poverkhnostei. (Theory of Surfaces). Gostekhizdat, 1956.

10. Sobolev, L.O. O dvizhenii simmetrichnogo volchka s polost'iu, napolnennoi zhidkost'iu. (On the motion of a symmetric top with a cavity filled with fluid). PMTF, No. 3, 1960.

11. Okhotsimskii, D.E. K teorii dvizhenia tela c polostiami, chastichno zapolnennymi zhidkost'iu. (On the theory of motion of a body with cavities partially filled with fluid). *PMM*, Vol. 20, No. 1, 1956.

12. Liusternik, L.A., Akushskii, I.Ia. and Ditkin, V.A. Tablitsy besselevykh funktsii. (Tables of Bessel's Functions). Gostekhizdat, 1949.

13. Ianke, E. and Emde, F., Tablitsy funktsii c formulami i krivymi. (Tables of Functions with Formulas and Curves). Fizmatgiz, 1959.

14. Gantmakher, F.R. Lektsii po analiticheskoi mekhanike (Lectures on Analytical Mechanics)., Fizmatgiz, 1960.

Translated by J.J.D.